A lattice gas of prime numbers and the Riemann Hypothesis

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Abstract

In recent years there has been some interest in applying ideas and methods taken from Physics in order to approach several challenging mathematical problems, particularly the Riemann Hypothesis, perhaps motived by the apparent inaccessibility to their solution from a full rigorous mathematical point of view. Most of these kind of contributions are suggested by some quantum statistical physics problems or by questions originated in chaos theory. In this note, starting from a very simple model of one-dimensional lattice gas and using the concept of equilibrium states as being described by Gibbs measures, we link classical statistical mechanics to the Riemann Hypothesis.

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The link between physics and number theory[1],[2] has been enriched in the few last years, particularly because of the possibility of enlighten, from a different point of view, some challenging mathematical problems, such as Riemann Hypothesis[3]. Most of the contributions in this sense come from the quantum side of statistical physics and chaos theory (see, for example, [4] and references therein). In this note we approach to the Riemann Hypothesis from a classical statistical mechanics model through the concept of equilibrium states as derived from a finite system variational principle.

Physically our model is very simple: a one-dimensional lattice gas in the grand canonical ensemble. The lattice is an interval of the natural numbers: $[1, M] \subset \mathbb{N}$ where M is large enough (eventually, in the thermodynamic limit, $M \to \infty$). The system is in contact with a particles reservoir characterized by the chemical potential μ and a heat reservoir at temperature T. For N particles the configuration of the system is given by $\omega \equiv (\omega_N, N)$ where $\omega_N \equiv (i_1, i_2, \cdots, i_N)$. The coordinates i_α $(\alpha = 1, 2, \cdots, N)$ take values in $[1, M]$ and N ranges, in principle, between 0 and ∞. The set of such configurations, the configuration space, will be denoted Λ.

We assume that each site in the lattice can have at most one particle, so we are considering a hard point pair potential

$$u_2\left(i_\alpha, i_\beta\right) = \begin{cases} \infty & \text{if } i_\alpha = i_\beta \\ 0 & \text{if } i_\alpha \neq i_\beta . \end{cases} \tag{1}$$

Also we assume that the particles are subject to a one-point potential such that they can just occupy sites in the lattice which are prime numbers:

$$u_1\left(i_\alpha\right) = \begin{cases} 0 \text{ if } i_\alpha \text{ is prime} \\ \infty \text{ if } i_\alpha \text{ is composite.} \end{cases} \tag{2}$$

By the way, an example of explicit function with this form is[5]: $u_1\left(i_\alpha\right) = -\log\left[\sin^2\left\{\pi\left[\left(i_\alpha - 1\right)!\right]^2 / i_\alpha\right\} / \sin^2\left\{\pi / i_\alpha\right\}\right]$. The energy for the configuration ω is $u\left(\omega\right) = \sum_{\alpha=1}^{N} u_1\left(i_\alpha\right) + \sum_{\alpha<\beta}^{N} u_2\left(i_\alpha, i_\beta\right)$.

Mathematically, a state of the system is a probability vector $\nu = \left(\nu\left(\omega\right) | \omega \in \Lambda\right)$. The set of all states is denoted by $\mathcal{M}$. In the state ν the system has the mean energy $\nu\left(u\right) := \sum_{\omega \in \Lambda} \nu\left(\omega\right) u\left(\omega\right)$. The (grand) partition function of u is defined $\Xi\left(\beta, \mu\right) := \sum_{\omega \in \Lambda} \exp\left[-\beta\left(u\left(\omega\right) - \mu N\right)\right]$ where $\beta = \left(k_B T\right)^{-1}$ with k_B the Boltzmann constant. For the parameters β, M and μ the Gibbs measure is defined

$$\nu_0(\omega) := \frac{1}{\Xi(\beta, M, \mu)} \exp\left[-\beta\left(u(\omega) - \mu N\right)\right]. \tag{3}$$

Given a measure ν we have the entropy $H(\nu) := -\sum_{\omega\in\Lambda} \nu(\omega) \log \nu(\omega)$ and the grand potential

$$\Omega[\nu] := \nu(u) - \frac{1}{\beta} H(\nu) - \mu\nu(N) \tag{4}$$

with $\nu(N) := \sum_{\omega\in\Lambda} \nu(\omega) N$.

To introduce the notion of equilibrium state we consider a variational principle for finite systems ($|\Lambda|$=finite) according to which, for given energy u and parameters β, M and μ, the Gibbs measure satisfies

$$\Omega[\nu_0] = -\frac{1}{\beta} \log \Xi(\beta, M, \mu) = \inf_{\nu\in\mathcal{M}} \left(\nu(u) - \frac{1}{\beta} H(\nu) - \mu\nu(N)\right). \tag{5}$$

A measure that attains this infimum is called an equilibrium state. Gibbs measure ν_0 is thus an equilibrium state. The result given by Eq.(5) is easily demonstrated[6] by using Jensen inequality applied to the concave function $x \mapsto \ln x$. The principle can be expressed saying that for any measure ν is $\Omega[\nu] \geqslant \Omega[\nu_0] = -\frac{1}{\beta} \log \Xi(\beta, M, \mu)$ with equality if and only if $\nu = \nu_0$.

Turning to our model, the grand potential for the Gibbs state is easily calculated if we explicitly write the grand partition function ($z = \exp[\beta\mu]$):

$$\Xi(\beta, M, z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \cdots \sum_{i_N=1}^{M} \exp\left[-\beta\left(\sum_{\alpha=1}^{N} u_1(i_\alpha) + \sum_{\alpha<\beta}^{N} u_2(i_\alpha, i_\beta)\right)\right]$$

and observe that, because of the limitation of occupation to just the prime numbers (Eq.2), the coordinates i_α ($\alpha = 1, 2, \cdots, N$) can take values only among the $\pi(M)$ prime numbers that exist in the interval $[1, M]$. Also, because of the impenetrability of the particles (Eq.1), the choice can be made of $\pi(M)!/[\pi(M) - N]!$ manners. We obtain

$$\Xi(\beta, M, z) = \sum_{N=0}^{\pi(M)} \binom{\pi(M)}{N} z^N = (1+z)^{\pi(M)} \tag{6}$$

and

$$\Omega[\nu_0] = -\frac{1}{\beta} \log(1+z)^{\pi(M)}. \tag{7}$$

The prime-counting function $\pi(x)$ is the number of prime numbers less or equal than x. In his 1859 classic memoir to the Berlin Academy of Sciences (see for example the monograph by Edwards[3] for a translation), Riemann gave an analytical formula for $\pi(x)$ in terms of the zeros and the pole of $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, the zeta function named after him. Restricting to values of x such that $x \geq 2^n$ with $n = 1, 2, ..$, it can be written[3]:

$$\pi(x) = \overline{\pi}(x) + \widetilde{\pi}(x), \tag{8}$$

where the smooth part, to which contribute the pole at $s = 1$ and the integer zeros of $\zeta(s)$, is given by

$$\overline{\pi}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left[\mathrm{Li}\left(x^{1/n}\right) - \sum_{k=1}^{\infty} \mathrm{Li}\left(x^{-2k/n}\right) \right], \tag{9}$$

with Li(x) the logarithmic integral function and $\mu(n)$ the Möbius function. The complex zeros, on the other hand, contribute to the oscillatory part

$$\widetilde{\pi}(x) = -2\mathrm{Re} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{\alpha=1}^{\infty} \mathrm{Li}\left(x^{(\sigma_\alpha + it_\alpha)/n}\right). \tag{10}$$

For x large enough the non-oscillatory and oscillatory parts can be written, respectively, as

$$\overline{\pi}(x) \approx \mathrm{Li}(x) \tag{11}$$

and

$$\widetilde{\pi}(x) \approx -\frac{2}{\log x} \sum_{\alpha=1}^{\infty} \frac{x^{\sigma_\alpha}}{(\sigma_\alpha^2 + t_\alpha^2)} \left[\sigma_\alpha \cos(t_\alpha \log x) + t_\alpha \sin(t_\alpha \log x)\right]. \tag{12}$$

For simplicity, from now on we will consider this last situation; so M will be assumed large enough.

In Eqs.(10) and (12) the real numbers σ_α and t_α are the real and imaginary part, respectively, of the complex number $\rho_\alpha = \sigma_\alpha + it_\alpha$ that verifies $\zeta(\rho_\alpha) = 0$. The properties of the complex or non trivial zeros of the Riemann´s zeta function has been extensively studied[3]. For example it has been demonstrated that there are infinitely many complex zeros and that all of them lie inside the region $0 < \Re(s) < 1$. Also is well known that if $\xi(s) := \frac{s}{2}(s-1)\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$ then the non trivial zeros of $\zeta(s)$ are precisely the zeros of $\xi(s)$ and since $\xi(s) = \xi(1-s)$, we have that the complex zeros of $\zeta(s)$ are symmetric with

respect to the so called critical line $\Re(s) = 1/2$. The Riemann Hypothesis is the statement that all the complex zeros have their real part exactly on the critical line: $\sigma_\alpha = 1/2\ \forall\alpha$. Although billions of complex zeros have been numerically calculated[7] confirming all of them this conjecture, the full demonstration of the general validity of the Riemann Hypothesis remains still open.

We wish to analyze the behavior of the equilibrium grand partition potential (Eq.7), with $\pi(M)$ given by Eqs.(8-12), as a function of the quantities $\sigma_\alpha \in (0,1)$ $(\alpha = 1, 2, \cdots)$. These numbers must be thought as possible values for the real part of the zeta function zeros and our goal would be to identify the true ones.

We focus into a generic zero ρ_γ and observe that, since Riemann actually didn't know the reliable values of the complex zeros, Eqs. (10) and (12) must be taken as general enough as to contemplate the possibility that, associated to $t_\gamma = \Im(\rho_\gamma)$, both cases, $\sigma_\gamma \neq 1 - \sigma_\gamma$ and $\sigma_\gamma = 1 - \sigma_\gamma$, can be considered for $\Re(\rho_\gamma)$.

Explicitly we rewrite Eq.(7) for β, M and z fixed

$$\Omega(\beta, M, z; \sigma_\gamma) = -\frac{1}{\beta}\log(1+z)^{\pi(M;\sigma_\gamma)}, \tag{13}$$

with

$$\pi(M;\sigma_\gamma) = \overline{\pi}(M) + \widetilde{\pi}_{\neq\gamma}(M) + \widetilde{\pi}_\gamma(M;\sigma_\gamma). \tag{14}$$

Here the term $\widetilde{\pi}_{\neq\gamma}(M)$ includes all the complex zeros except that labelled γ and we assume that its contribution to the prime-counting function is the true one. We write the remaining term:

$$\widetilde{\pi}_\gamma(M;\sigma_\gamma) \approx \begin{cases} -\frac{2}{\log M}\left(\frac{M^{\sigma_\gamma}}{\sigma_\gamma^2+t_\gamma^2}\left[\sigma_\gamma\cos(t_\gamma\log M) + t_\gamma\sin(t_\gamma\log M)\right] + \right. \\ \left. \frac{M^{(1-\sigma_\gamma)}}{(1-\sigma_\gamma)^2+t_\gamma^2}\left[(1-\sigma_\gamma)\cos(t_\gamma\log M) + t_\gamma\sin(t_\gamma\log M)\right]\right) & \text{for } \sigma_\gamma \neq 1-\sigma_\gamma \\ -\frac{2}{\log M}\frac{M^{1/2}}{\frac{1}{4}+t_\gamma^2}\left[\frac{1}{2}\cos(t_\gamma\log M) + t_\gamma\sin(t_\gamma\log M)\right] & \text{for } \sigma_\gamma = 1-\sigma_\gamma, \end{cases} \tag{15}$$

where the fact that if $\zeta(\sigma_\gamma + it_\gamma) = 0$ then also is $\zeta(1-\sigma_\gamma + it_\gamma) = 0$, has been considered.

Special cases can make the analysis even simpler. For $M = M_1 \equiv \exp[\frac{\pi}{2}(4n-1)/|t_\gamma|]$

and $M = M_2 \equiv \exp[\frac{\pi}{2}(4n-3)/|t_\gamma|]$ ($n \in \mathbb{N}$ large enough) and taking into account that $|t_\alpha| \geq 14.37$ for all the non trivial zeros of the zeta function, we have

$$\widetilde{\pi}_\gamma(M_{1,2};\sigma_\gamma) \approx \begin{cases} \pm\frac{2}{|t_\gamma|\log M_{1,2}}\left(M_{1,2}^{\sigma_\gamma} + M_{1,2}^{(1-\sigma_\gamma)}\right) & \text{for } \sigma_\gamma \neq 1-\sigma_\gamma \\ \pm\frac{2}{|t_\gamma|\log M_{1,2}}M_{1,2}^{1/2} & \text{for } \sigma_\gamma = 1-\sigma_\gamma, \end{cases} \tag{16}$$

where the upper sign corresponds to M_1. Note the avoidable discontinuity in the function $\widetilde{\pi}_\gamma(M;\sigma_\gamma)$ when $\sigma_\gamma = 1-\sigma_\gamma$.

Using Eq.(13) together with Eq.(15) -or the particular cases given by Eq.(16)- is easy to see that the grand potential has a unique extremum (minimum or maximum depending on M) at the interval $0 < \sigma_\gamma < 1$. From a heuristic point of view it is reasonable to expect that this extremum (maximum = unstable equilibrium; minimum = stable equilibrium) be reached when σ_γ takes the value $(\sigma_\gamma)_{true}$ that gives, through Eq.(14), the correct number of prime numbers lying inside the interval $[1,M]$. The same conclusion can be achieved in a more formal way if we take into account that fixing a given value for σ_γ can be thought as imposing a constraint that limits (through $\pi(M;\sigma_\gamma)$) the number of the accessible states to the system. If this constraint is removed, so σ_γ is leaved to freely vary in the interval $(0,1)$, then the equilibrium probability $P(\sigma_\gamma)$ that the system be in states with the parameter taking values in the interval between σ_γ and $\sigma_\gamma + \delta\sigma_\gamma$ behaves as[8]

$$P(\sigma_\gamma) \propto \begin{cases} \exp[-\beta\Omega(\beta,M,z;\sigma_\gamma)] & \text{if } \sigma_\gamma \in (0,1) \\ 0 & \text{if } \sigma_\gamma \notin (0,1). \end{cases} \tag{17}$$

Moreover, σ_γ can be treated as a stochastic variable $\sigma_\gamma = \sigma_\gamma(t)$ describing, for example, the position of an hypothetical Brownian particle of mass m moving in a force field with potential function given by

$$U(\sigma_\gamma) = \begin{cases} \Omega(\beta,M,z;\sigma_\gamma) & \text{if } \sigma_\gamma \in (0,1) \\ \infty & \text{if } \sigma_\gamma \notin (0,1). \end{cases} \tag{18}$$

In this picture, $P(\sigma_\gamma)$ is the marginal of the joint probability distribution

$$P(\sigma_\gamma,\dot{\sigma_\gamma}) = K\exp\left\{-\beta\left[U(\sigma_\gamma) + \frac{1}{2}m\dot{\sigma_\gamma}^2\right]\right\}, \tag{19}$$

where $\dot{\sigma_\gamma} \equiv \frac{d\sigma_\gamma}{dt}$ and K is the normalization constant. By direct substitution one can verify that this probability distribution is the stationary solution of the Fokker-Planck equation[9]

$$\frac{\partial}{\partial t}P\left(\sigma_\gamma,\dot{\sigma_\gamma},t\right)=\mathcal{L}\left(\sigma_\gamma,\dot{\sigma_\gamma},\partial_{\sigma_\gamma},\partial_{\dot{\sigma_\gamma}}\right)P\left(\sigma_\gamma,\dot{\sigma_\gamma},t\right)$$
$$\equiv\left\{-\sigma_\gamma\frac{\partial}{\partial\sigma_\gamma}+\frac{1}{m}\frac{\partial}{\partial\dot{\sigma_\gamma}}\left[\frac{dU\left(\sigma_\gamma\right)}{d\sigma_\gamma}+\lambda\dot{\sigma_\gamma}\right]+\frac{\lambda}{\beta m^2}\frac{\partial^2}{\partial\dot{\sigma_\gamma}^2}\right\}P\left(\sigma_\gamma,\dot{\sigma_\gamma},t\right),\quad(20)$$

where λ is a friction coefficient.

Associated to the Fokker-Planck equation (20) with the initial condition $P(\sigma_\gamma,\dot{\sigma_\gamma},t_0)=\delta\left(\sigma_\gamma-\sigma_{\gamma0}\right)\delta(\dot{\sigma_\gamma}-\dot{\sigma}_{\gamma0})$ we have[10] the Langevin stochastic differential equation

$$m\frac{d\dot{\sigma_\gamma}}{dt}=-\frac{dU\left(\sigma_\gamma\right)}{d\sigma_\gamma}-\lambda\dot{\sigma_\gamma}+\sqrt{\frac{2\lambda}{\beta}}f\left(t\right),\quad(21)$$

where $f\left(t\right)$ is an additive Gaussian stochastic process δ-correlationated and of zero mean. Here $-\lambda\dot{\sigma_\gamma}$ represents the friction force that affects the Brownian particle. Given the arbitrariness (to our purpose) of the friction parameter, we simplify the analysis by choosing as such $\lambda=0$ so that Eq.(21) yields the simpler Newton deterministic equation

$$m\frac{d\dot{\sigma_\gamma}}{dt}=-\frac{dU\left(\sigma_\gamma\right)}{d\sigma_\gamma}.\quad(22)$$

If we assume that initially the particle is placed at the position $\sigma_{\gamma_0}=\left(\sigma_\gamma\right)_{extr}$ that extremises $\Omega\left(\beta,M,z;\sigma_\gamma\right)$ with velocity $\dot{\sigma}_{\gamma_0}=0$, then the equation of motion has as the unique obvious solution the isolated fixed point $\sigma_\gamma\left(t\right)=\left(\sigma_\gamma\right)_{extr}\,\forall t$. Taking into account that for any other pair of initial conditions the solution is time-dependent and that in the mathematical world the zeros of $\zeta\left(s\right)$ do not change with time but remain constant forever, we deduce that it must be $\left(\sigma_\gamma\right)_{true}\equiv\left(\sigma_\gamma\right)_{extr}$. One then infers that $\left(\sigma_\gamma\right)_{true}$ should verify

$$\left.\frac{\partial\Omega\left(\beta,M,z;\sigma_\gamma\right)}{\partial\sigma_\gamma}\right|_{\sigma_\gamma=\left(\sigma_\gamma\right)_{true}}=-\frac{1}{\beta}\log\left(1+z\right)\left.\frac{\partial\widetilde{\pi}_\gamma\left(M;\sigma_\gamma\right)}{\partial\sigma_\gamma}\right|_{\sigma_\gamma=\left(\sigma_\gamma\right)_{true}}=0.\quad(23)$$

The symmetry with respect to σ_γ and $1-\sigma_\gamma$ of expressions (15) or (16) implies that this equation has the form $\left[f\left(\sigma_\gamma\right)-f\left(1-\sigma_\gamma\right)\right]_{\sigma_\gamma=\left(\sigma_\gamma\right)_{true}}=0$ (with $f\left(\sigma_\gamma\right)\neq0$; $f\left(1-\sigma_\gamma\right)\neq0$) which is clearly verified by $\left(\sigma_\gamma\right)_{true}=1-\left(\sigma_\gamma\right)_{true}=1/2$. Because the zero ρ_γ that we have considered is a generic one, we would conclude that $\left(\sigma_\alpha\right)_{true}=1/2\,\forall\alpha$, say that, in fact, the real part of all non trivial zeros of Riemann's zeta function lie on the critical line.

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